Symmetry Reduction in the Superpattern (SP) Problem

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Permutations and Pattern Containment

A permutation of [n]**:** A bijection $\pi : [n] \to [n]$ written as,

- One line form: $\pi = \pi(1)\pi(2)\ldots\pi(n)$
- Matrix form: $A_\pi(j,i) = \begin{cases} 1 & \text{if } \pi(i) = j \\ 0 & \text{if } \pi(i) \neq j \end{cases}$

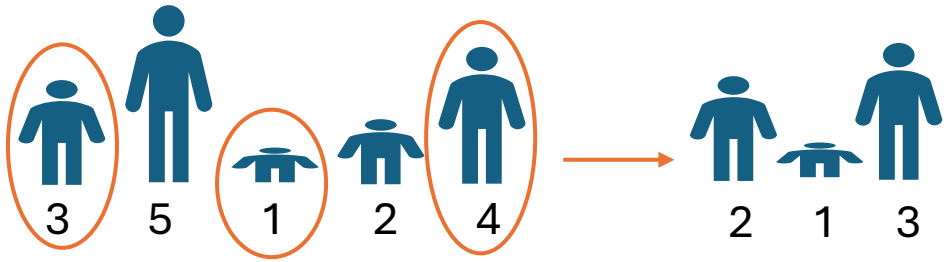
Example:

$$\pi = 35124 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

π contains σ:

σ occurs in π iff some k positions in π are order − isomorphic to σ

- π = 35124 contain σ = 213 as a pattern?
  Yes



The Superpattern Problem

$\pi \in S_n$ **is a k-superpattern (kSP)** if every $\sigma \in S_k$ occurs in π.

$$m(k) = min\{n : \exists \text{ k-superpattern in } S_n\}$$

Example: n=5, k=3

π	2	5	3	1	4
σ_1	1		2		3
σ_2	1	3		2	
σ_3			2	1	3
σ_4	2	3		1	
σ_5		3	2	1	
σ_6		3		1	2

Important considerations:

- A kSP must fit all k! subsequences
- A permutation of length n has $\binom{n}{k}$ subsequences.
- $\binom{n}{k} \geq k! \Rightarrow n \geq \frac{k^2}{e^2}$

Bounds on Minimal Superpatterns

Trivial bounds: $k \leq m(k) \leq k! \cdot k$

Counting + Engen-Vatter Construction (current best)**:**

$$k^2\left(\tfrac{1}{e^2} + o(1)\right) \leq m(k) \leq \left\lfloor \frac{k^2+1}{2} \right\rfloor$$

Must contain at least k! subsequences

Concatenate ascending and descending runs to form a permutation that must contain all patterns.

- Neither bound is sharp
- The EV disregards subclass structure.

EV Example:

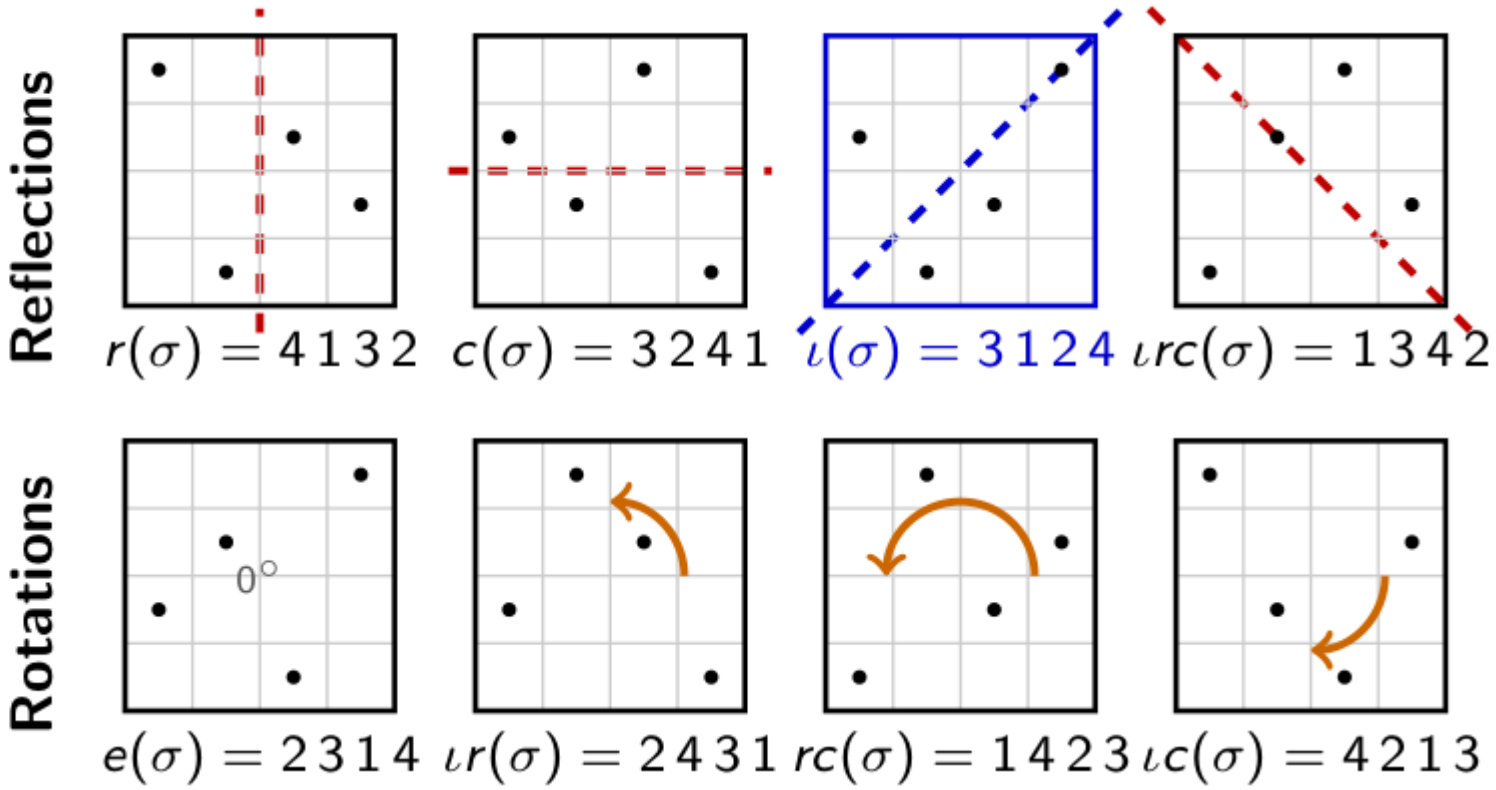
N=3: 13 2 13 ⇒ **25314**

Symmetries: D_8 on permutations

Plot $\pi \in S_n$ as dots on a $n \times n$ grid for $(i, \pi(i))$.

- The 8 symmetries of the square act on the plot.

$$D_8 = \langle r, c, \iota \rangle$$



Key Observations:

σ is a k-SP

$\Longleftrightarrow$

$g(\sigma)$ is a k-SP $g \in D_8$

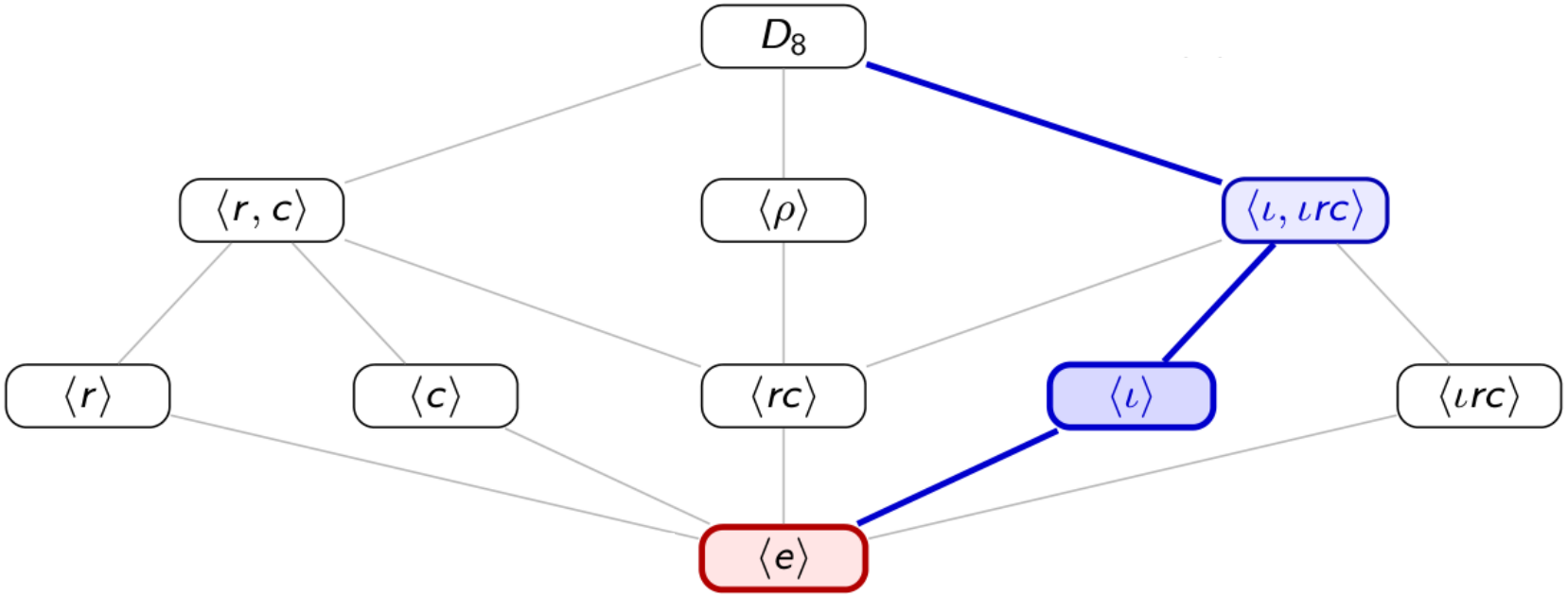
Remember:

- ι = reflection across the main diagonal (transpose)

A Lattice of Relaxations

For each subgroup $G \leq \mathrm{D}_8$

$a_G(k) =$ min length π: π contains a pattern representative for each G−orbit on S_k



Monotonicity. $G \leq H \implies a_H(k) \leq a_G(k)$. More symmetry ⇒ fewer orbits ⇒ shorter covering.

Today's branch: $G = \langle \iota \rangle$. Orbits are involutions and inverse-pairs.

Not Quite There: Still $1/e^2$

Lower bound with symmetry: Covering $k!/|G|$ orbits,

$$\binom{n}{k} \geq k!/|G| \quad \Longrightarrow \quad n \geq \left(\frac{k!}{|G|}\right)^{2/k}$$

Stirling: $(k!)^{2/k} \sim k^2/e^2$, and $|G|^{-2/k} \to 1$ for any fixed G. So

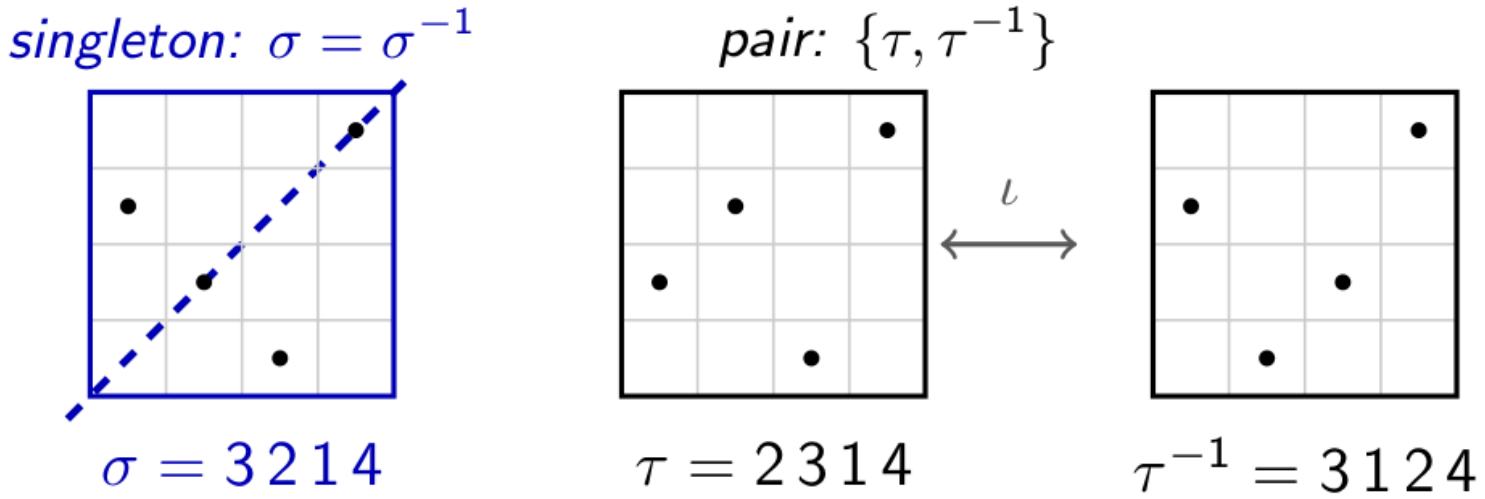
$$\boxed{a_G(k) \geq \left(\tfrac{1}{e^2} + o(1)\right) k^2 \quad \text{for every } G \leq D_8.}$$

The opening. If we want sub-k^2 savings, the lever isn't symmetry alone. It's structure inside a sub-class that EV's uniform construction ignores.

Involutions: the Fixed Points of ι

Orbits of $\langle \iota \rangle$ on S_k split into two types:

- **Singletons:** $\{\sigma\}$ with $\sigma = \sigma^{-1}$, involutions.
- **Pairs:** $\{\sigma, \sigma^{-1}\}$ for non-involutions.



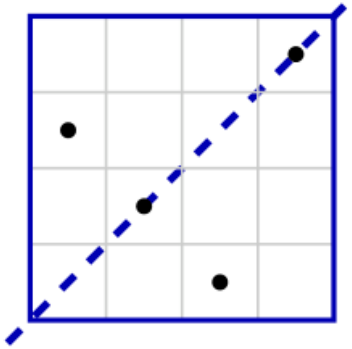
Count (Moser–Wyman / Knuth): $|I_k| \sim \dfrac{(k/e)^{k/2}\, e^{\sqrt{k}}}{(4\pi k)^{1/4}} \approx \sqrt{k!}\,.$

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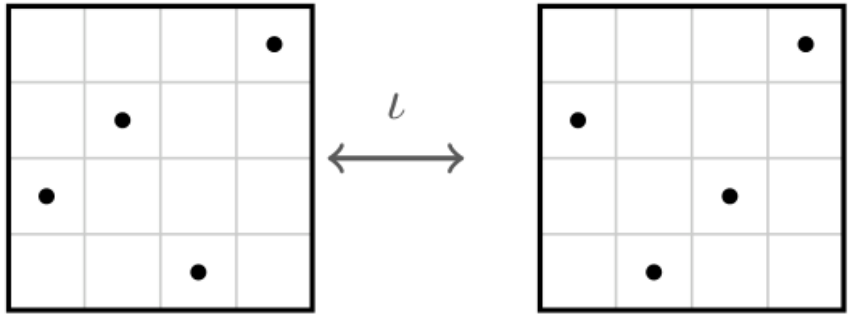
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singleton: $\sigma = \sigma^{-1}$



$\sigma = 3\,2\,1\,4$

pair: $\{\tau, \tau^{-1}\}$



$\tau = 2\,3\,1\,4$ $\qquad$ $\tau^{-1} = 3\,1\,2\,4$

Count (Moser–Wyman / Knuth):

$$|I_k| \sim \frac{(k/e)^{k/2}\, e^{\sqrt{k}}}{(4\pi k)^{1/4}} \approx \sqrt{k!}\,.$$

Decompose:

$$a_{\langle \iota \rangle}(k) \leq m_{\text{inv}}(k) + m^{\iota}_{\text{nonInv}}(k)$$

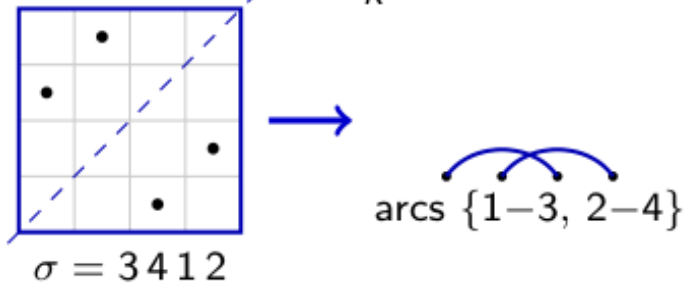
Min length containing all involutions.

Min length containing one rep per inverse-pair.

Why Involutions? Structures to Exploit

Counting says that sub-k^2 is possible. Two features say it might be reachable:

Translation. An involution $\sigma \in S_k$ *is* an ordered matching on $[k]$:



Bijection:
involution pattern containment = induced sub-matching containment

min n: an ordered matching on $[n]$ inducing every ordered matching on $[k]$.

Tools Unlocked:

- **RSK:** involutions ⇔ SYT ⟹ hook-length, jeu de taquin, shape-bounded, LIS/LDS.
- **Arc-diagrams:** crossing/nesting stats, non-crossing matchings, arc-by-arc recursion.
- **Inversion Closure:** patterns of an involution container come with inverses free.

The Sub-quadratic Bound

Counting on m_{inv}. Need $\binom{n}{k} \geq |I_k| \sim \sqrt{k!}$.

$$\frac{n^k}{k!} \geq \sqrt{k!} \quad \Longrightarrow \quad n^k \geq (k!)^{3/2} \quad \Longrightarrow \quad n \geq (k!)^{3/(2k)}$$

Stirling: $(k!)^{3/(2k)} \sim \left(\frac{k}{e}\right)^{3/2}$

$$\boxed{m_{\text{inv}}(k) \geq \left(\frac{1}{e^{3/2}} + o(1)\right) k^{3/2}.}$$

Only known sub-k^2 counting bound in this framework

Gap: Lower bound $k^{3/2}/e^{3/2}$, Upper bound $k^2/2$. Wide open.

Open Questions & Outlook

$$\text{Is } m_{\text{inv}}(k) = \Theta(k^{3/2})?$$

Lower bound $\Omega(k^{3/2})$ is counting. Upper bound: no sub-k^2 construction known.

- **Computation:** Compute $\mathrm{m_{inv}(k)}$ for $\mathrm{k} \leq 6$ directly. Does the $\mathrm{k}^{3/2}$ scale appear.
- **Non-involution half:** Bound $m^{\iota}_{\text{nonInv}}(k)$ - one rep per inverse-pair.
- **Other fixed-point classes:** $\langle r \rangle$-fixed = palindromes; $\langle c \rangle$-fixed = self-complements.
- **Modifying EV:** Adapt EV construction for $\mathrm{m_{inv}(k)}$. Perhaps $\mathrm{k}^{3/2}$.

Thank you. Questions?

References

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