Symmetry Reduction in the Superpattern Problem

Involutions and the Search for Sub-quadratic Savings

James Watt · Department of Mathematics, University of Florida

Advisor: Dr. Miklós Bóna

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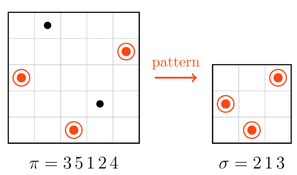
Abstract

A permutation is a **k-superpattern** if it contains every permutation of length k as a pattern. The minimal length $m(k)$ is known only to satisfy $k^2(1/e^2 + o(1)) \le m(k) \le \lceil (k^2+1)/2 \rceil$, and neither bound is sharp. We study **symmetry-reduced relaxations** $a_G(k)$, indexed by subgroups G of the dihedral group $\mathbf{D_8}$ acting on permutation plots, in which a container needs only one pattern from each G-orbit. We show that **symmetry alone cannot break the quadratic barrier**: counting still gives $a_G(k) \ge (1/e^2 + o(1))k^2$ for every G. Focusing on the inverse symmetry ι, whose orbits are **involutions** and inverse pairs, we split the problem in two. Because there are only about $\sqrt{k!}$ involutions, counting gives $m_{\text{inv}}(k) \ge (e^{-3/2} + o(1))k^{3/2}$, **leaving room for sub-quadratic behavior**, and the bijection between involutions and ordered matchings brings RSK and arc-diagram tools to the remaining gap.

Introduction

A **permutation** $\pi \in S_n$ is an ordering of 1, …, n.

π **contains** $\sigma \in S_k$ if some k entries of π appear in the same relative order as σ.



35124 contains 213 (entries 3, 1, 4).

π is a **k-superpattern** if it contains every $\sigma \in S_k$. We want the shortest one:

m(k) = the length of the shortest k-superpattern.

Example: 25314 contains all six patterns of length 3, so $m(3) \le 5$.

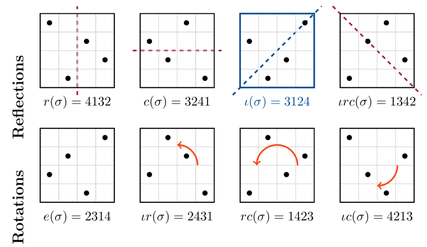
π	2	5	3	1	4
123	1		2		3
132	1	3	2		
213			2	1	3
231	2	3		1	
312		3		1	2
321		3	2	1	

Each row: one occurrence.

$$k^2\left(\frac{1}{e^2} + o(1)\right) \le m(k) \le \left\lceil \frac{k^2+1}{2} \right\rceil$$

Lower: a length-n permutation has only (n choose k) subsequences but must realize all k! patterns. **Upper:** Engen–Vatter[1] concatenate ascending and descending runs (k = 3: 13 2 13 → 25314). **Neither bound is sharp.**

Symmetry Reduction



$D_8 = \langle r, c, \iota \rangle$ (reverse, complement, inverse) acting on σ = 2314. Every symmetry preserves containment, so π is a k-superpattern ⟺ g(π) is, for every $g \in D_8$.

Symmetry alone is not enough. Covering k!/|G| orbits:

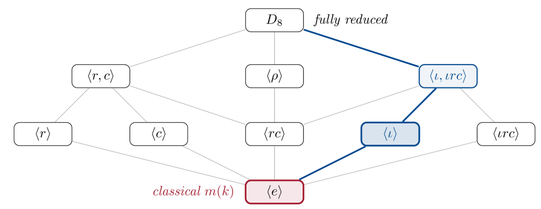
$$\frac{n^k}{k!} \ge \binom{n}{k} \ge \frac{k!}{|G|} \implies n \ge \left(\frac{(k!)^2}{|G|}\right)^{1/k}$$

$$\boxed{a_G(k) \ge \left(\frac{1}{e^2} + o(1)\right)k^2 \quad \text{for every } G \le D_8}$$

since $(k!)^{2/k} \sim k^2/e^2$ and $|G|^{-1/k} \to 1$. Sub-k^2 savings must come from **structure inside a subclass**.

For each subgroup $G \le D_8$, relax to one pattern per G-orbit:

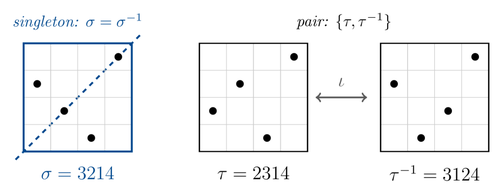
$$a_G(k) = \min\{ n : \exists \pi \in S_n \text{ containing a representative of every } G\text{-orbit on } S_k \}$$



*The subgroup lattice of D_8. ⟨e⟩ is the classical m(k); **monotone:** $G \le H \implies a_H(k) \le a_G(k)$.*

This work: G = ⟨ι⟩ (the highlighted path). Its orbits on S_k are **involutions** (σ = σ⁻¹, singletons) and **inverse pairs** {σ, σ⁻¹}.

The Involution Branch



Involutions have plots symmetric about the diagonal; every other σ is paired with its inverse by ι.

Involutions are rare (Moser–Wyman[4], Knuth[5]): only ≈ √k! of them

$$|I_k| \sim \frac{(k/e)^{k/2}\, e^{\sqrt{k}}}{(4e)^{1/4}}, \qquad |I_k|^{1/k} \sim \sqrt{k/e}$$

Decompose by orbit type (concatenate two containers):

$$a_{\langle \iota \rangle}(k) \le m_{\text{inv}}(k) + m^{\iota}_{\text{nonInv}}(k)$$

m_{inv}: shortest permutation containing every involution; $m^{\iota}_{\text{nonInv}}$: one representative of every inverse pair.

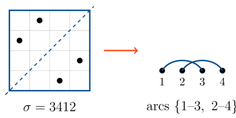
Counting on m_{inv}: need (n choose k) ≥ $|I_k|$.

$$\frac{n^k}{k!} \ge \binom{n}{k} \ge |I_k| \implies n^k \gtrsim (k!)^{3/2} \implies n \gtrsim \left(\frac{k}{e}\right)^{3/2}$$

$$\boxed{m_{\text{inv}}(k) \ge \left(\frac{1}{e^{3/2}} + o(1)\right) k^{3/2}}$$

The only sub-k^2 counting bound in this framework.

Involutions are ordered matchings, and involution containment is induced sub-matching containment:



- **RSK:** involutions ⟺ standard Young tableaux (hook lengths, LIS/LDS)
- **Arc diagrams:** crossing/nesting statistics, arc-by-arc recursion
- **Inverse closure:** an involution container gets inverses free

Conclusion

- D_8 gives a **lattice of relaxations** $a_G(k)$, from the classical m(k) to the fully reduced problem.
- **Symmetry alone cannot beat k^2**: every $a_G(k) \ge (1/e^2 + o(1))k^2$.
- Savings must come from **structure inside a subclass**. For involutions, counting allows order $\mathbf{k^{3/2}}$, far below $k^2/2$.
- Involutions are **ordered matchings**, bringing RSK and arc-diagram tools to bear on the gap.

The gap for involutions:

$$\left(\frac{1}{e^{3/2}} + o(1)\right)k^{3/2} \le m_{\text{inv}}(k) \le \left\lceil \frac{k^2+1}{2} \right\rceil$$

Open Questions

$$\text{Is } m_{\text{inv}}(k) = \Theta\left(k^{3/2}\right) ?$$

- **Computation:** find $m_{\text{inv}}(k)$ for $k \le 6$. Does the $k^{3/2}$ scale appear?
- **Non-involution half:** bound $m^{\iota}_{\text{nonInv}}(k)$.
- **Other fixed-point classes:** permutations fixed by rc or ιrc, also ≈ √k! in number.
- **Modify Engen–Vatter** for involutions, perhaps reaching $k^{3/2}$.

Acknowledgements

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References

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